Math 1020 Week 4

Trigonometry

We measure angles using an unit called:

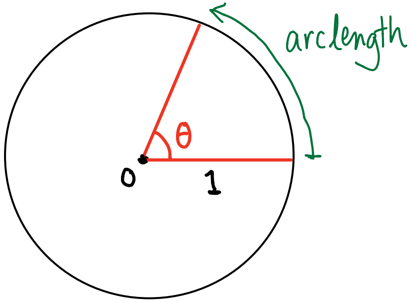
$\boxed{\text{Radian}}$

Consider a unit circle (radius = 1)

Define

$\theta = x$ rad

if arclength $= x$



360° = full circle $= 2\pi$ rad

180° = half circle $= \pi$ rad

90° = right angle $= \frac{\pi}{2}$ rad

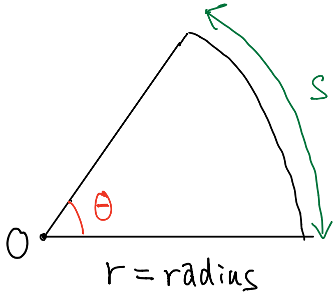
Conversion:

$$\text{Degree} \underset{\times \frac{180}{\pi}}{\overset{\times \frac{\pi}{180}}{\rightleftarrows}} \text{Radian}$$

$$x^\circ = \frac{\pi x}{180} \text{ rad} \qquad y \text{ rad} = \frac{180 y}{\pi}^\circ$$

Formulas

For a sector

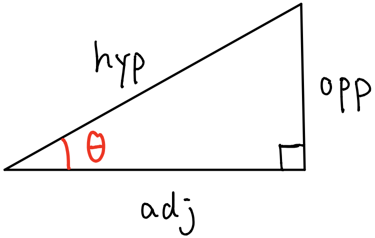


Arclength: $s = r\theta$

Area: $A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$

Trigonometric Functions: 1st definition

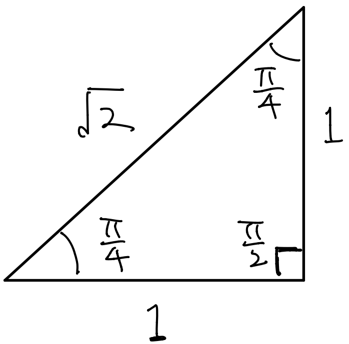
For $0 \leq \theta \leq \frac{\pi}{2}$

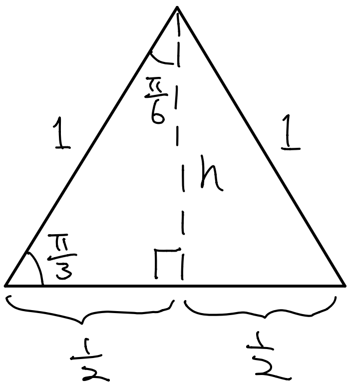


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Values at special angles

θ in deg	30°	45°	60°
θ in rad	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$



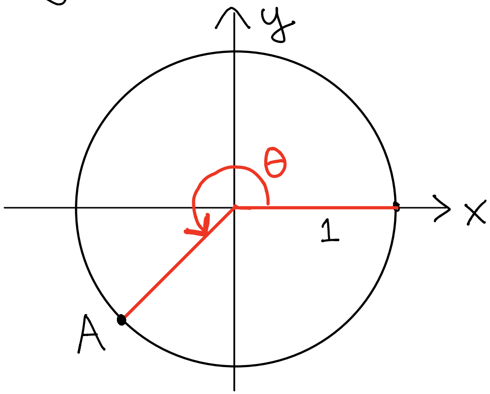


equilateral △

$$h = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$

Extension of $\sin x$ and $\cos x$ to $\mathbb{R}$

Consider the unit circle centered at the origin

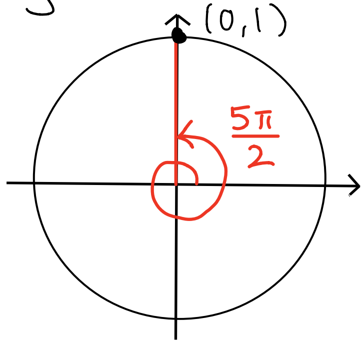


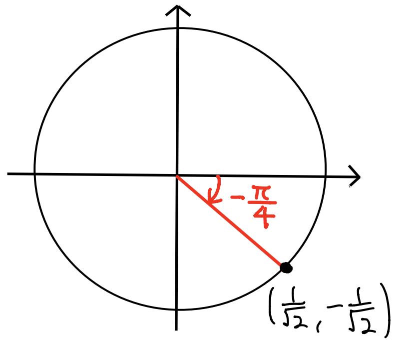
θ is measured anti-clockwisely from positive x-axis

Defn

$A = (\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$

eg



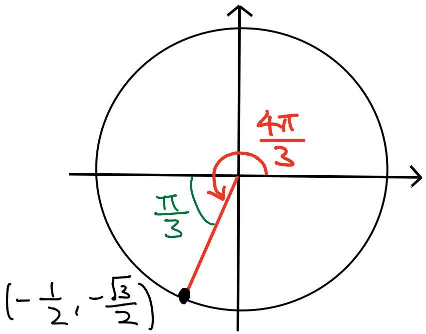


$$\cos\frac{5\pi}{2} = 0$$

$$\sin\frac{5\pi}{2} = 1$$

$$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

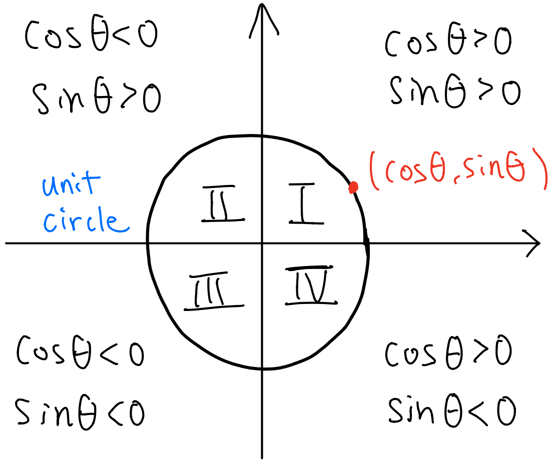
$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



$$\cos\frac{4\pi}{3} = -\frac{1}{2}$$

$$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

Sign of $\sin\theta$, $\cos\theta$



"CAST" diagram

For $\sin\theta$, $\cos\theta$, $\tan\theta$ in each quadrant

Only $\sin\theta>0$	S	A	All >0
Only $\tan\theta>0$	T	C	Only $\cos\theta>0$

More definitions:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Rmk

① For domains, note that

$$\sin\theta = 0 \text{ if } \theta = 0, \pm\pi, \pm 2\pi, \cdots$$

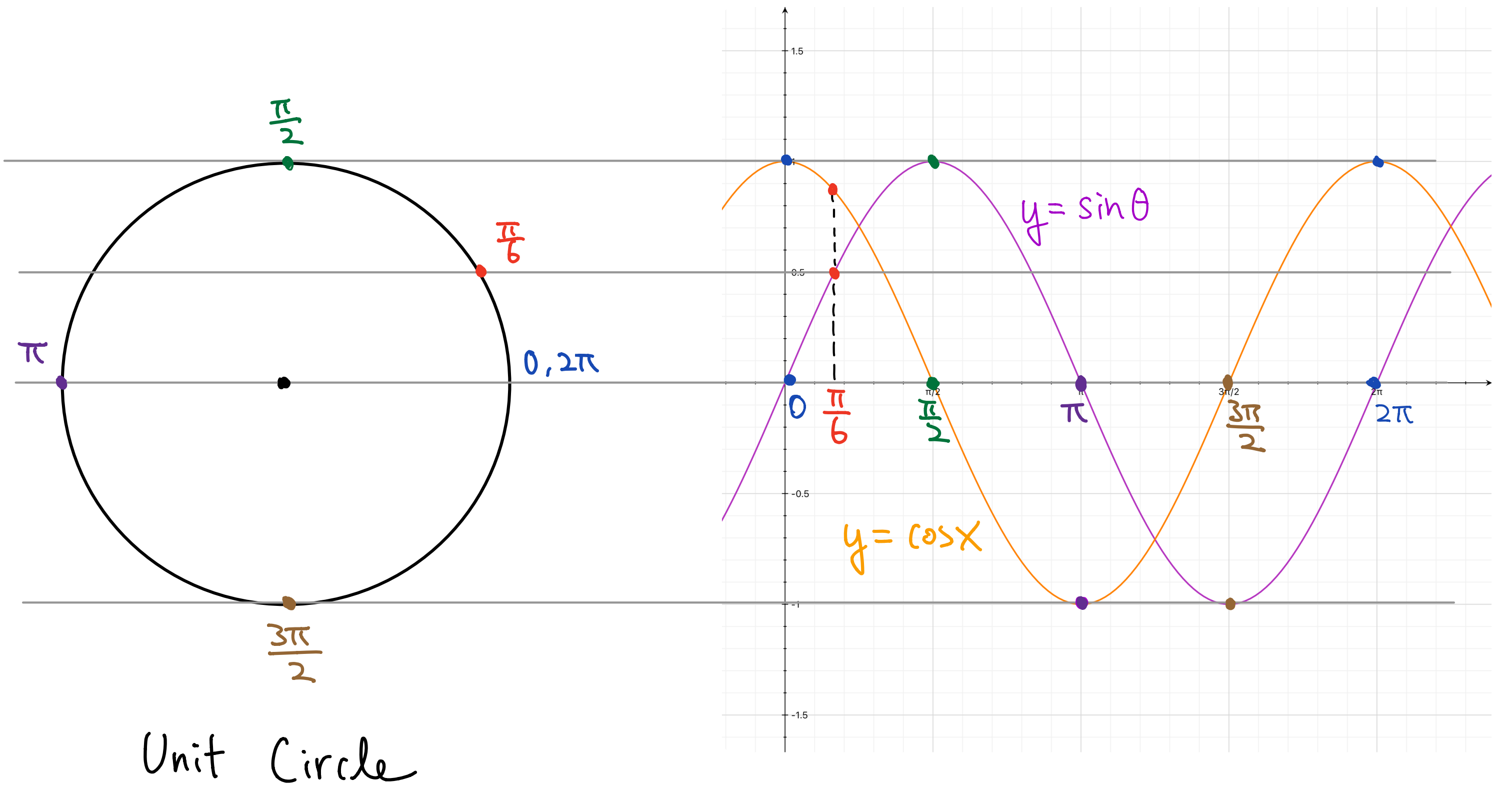
$$\cos\theta = 0 \text{ if } \theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$$

$$\therefore D_{\cot\theta} = D_{\csc\theta} = \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$$

$$D_{\tan\theta} = D_{\sec\theta} = \mathbb{R} \setminus \{(k+\tfrac{1}{2})\pi, k \in \mathbb{Z}\}$$

② $\cot\theta = \dfrac{1}{\tan\theta}$ when $\theta \neq \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$

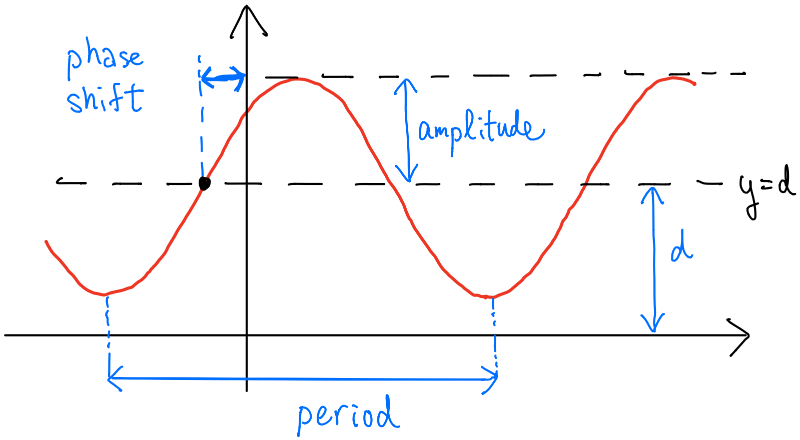
Graph of sinx and cosx



Variation

$$y = a\sin(bx + c) + d \qquad a, b > 0$$

$$= a\sin\left[b\left(x + \frac{c}{b}\right)\right] + d$$



center line $y = d$ period $= \frac{2\pi}{b}$

amplitude $= a$ phase shift $= -\frac{c}{b}$

eg Graph $y = 3\cos(2x+1) + 4$

Sol

$\cos x$

$\downarrow$ $\leftarrow$ 1 unit

$\cos(x+1)$

$\downarrow$ $\rightarrow\leftarrow$ half $= \frac{1}{2}$

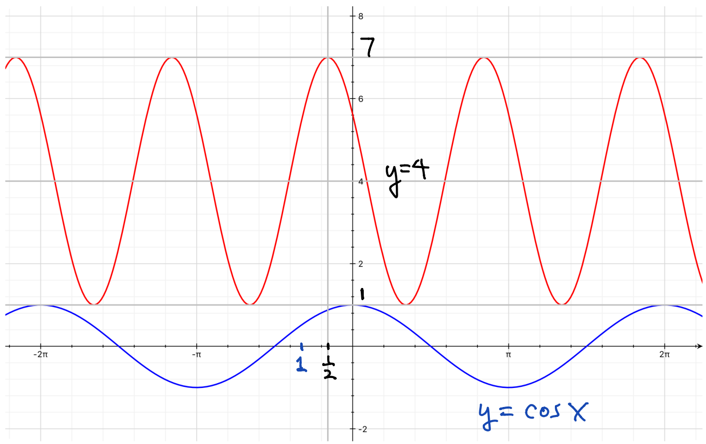
$\cos(2x+1)$

$\downarrow$ $\updownarrow$ 3 times

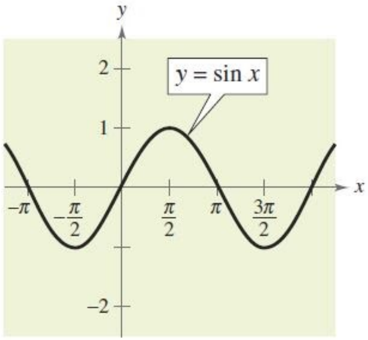
$3\cos(2x+1)$

$\downarrow$ $\uparrow$ 4 unit

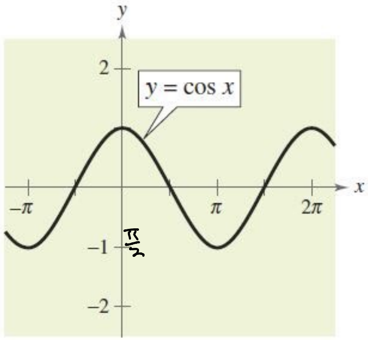
$3\cos(2x+1) + 4$



Graphs of the basic Trig. functions



DOMAIN: $(-\infty, \infty)$
RANGE: $[-1, 1]$
PERIOD: 2π



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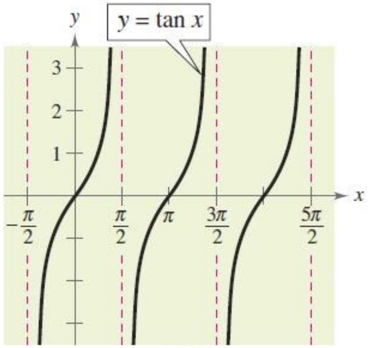
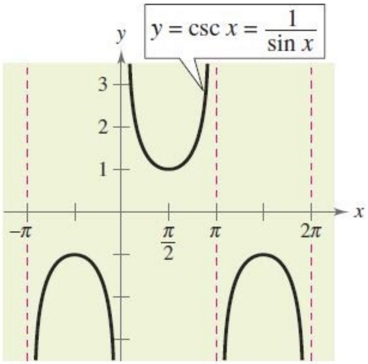
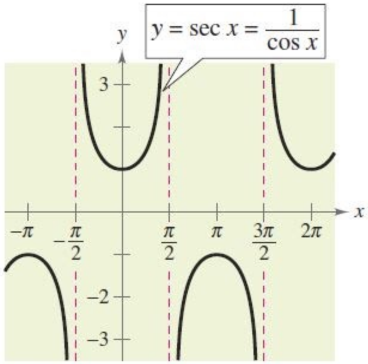


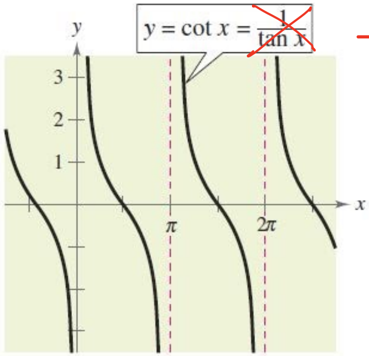
DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π



DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$
RANGE: $(-\infty, -1] \cup [1, \infty)$
PERIOD: 2π



$\frac{\cos x}{\sin x}$

DOMAIN: ALL $x \neq n\pi$
RANGE: $(-\infty, \infty)$
PERIOD: π

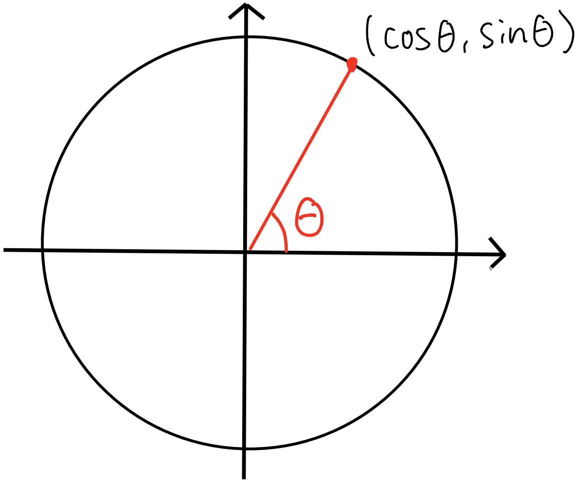
Trigonometric Identity

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$ ... 1

$\tan^2\theta + 1 = \sec^2\theta$ ... 2

$1 + \cot^2\theta = \csc^2\theta$ ... 3



Pf of 1

From our defn, $(\cos\theta, \sin\theta)$ is on the unit circle

$\Rightarrow \cos^2\theta + \sin^2\theta = 1 \Rightarrow$ 1

Pf of 2

$$\begin{aligned}
\text{L.H.S.} &= \tan^2\theta + 1 \\
&= \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} \\
&= \left(\frac{1}{\cos\theta}\right)^2 \\
&= \sec^2\theta \\
&= \text{R.H.S.}
\end{aligned}$$

Pf of 3 is similar

eg Let θ be in quadrant II and $\sin\theta = \frac{1}{3}$

Find $\cos\theta$ and $\cot\theta$.

Sol

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \left(\frac{1}{3}\right)^2 + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{8}{9}$$

θ is quadrant II $\Rightarrow \cos\theta < 0$

S	A
T	C

$$\therefore \cos\theta = -\frac{\sqrt{8}}{3} = -\frac{2\sqrt{2}}{3}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}$$

eg Show that

$$\frac{\sec\theta - 1}{\tan\theta} = \frac{\tan\theta}{\sec\theta + 1}$$

Sol

$$\text{L.H.S.} = \frac{\sec\theta - 1}{\tan\theta} \cdot \frac{\sec\theta + 1}{\sec\theta + 1}$$

$$= \frac{\sec^2\theta - 1}{\tan\theta(\sec\theta + 1)}$$

$$= \frac{\tan^2\theta}{\tan\theta(\sec\theta + 1)}$$

$$= \frac{\tan\theta}{\sec\theta + 1}$$

$$= \text{R.H.S.}$$

Even function: $f(-x) = f(x)\ \forall x$

Odd function: $f(-x) = -f(x)\ \forall x$

Even/Odd Formulas

$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$

Cofunction Formulas

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$$

Periodic Formulas

If n is an integer.

$\sin(\theta + 2\pi n) = \sin\theta \qquad \csc(\theta + 2\pi n) = \csc\theta$

$\cos(\theta + 2\pi n) = \cos\theta \qquad \sec(\theta + 2\pi n) = \sec\theta$

$\tan(\theta + \pi n) = \tan\theta \qquad \cot(\theta + \pi n) = \cot\theta$

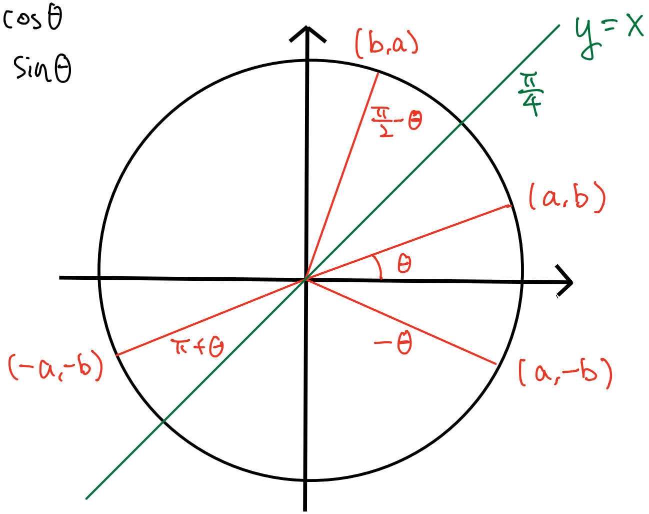
$\sin(\theta + \pi) = -\sin\theta$

$\cos(\theta + \pi) = -\cos\theta$

How to get these formulas?

If $a = \cos\theta$

$b = \sin\theta$



eg

$(\cos(-\theta), \sin(-\theta)) = (a, -b) = (\cos\theta, -\sin\theta)$

$\left(\cos\left(\frac{\pi}{2}-\theta\right), \sin\left(\frac{\pi}{2}-\theta\right)\right) = (b, a) = (\sin\theta, \cos\theta)$

$(\cos(\theta+\pi), \sin(\theta+\pi)) = (-a, -b) = (-\cos\theta, -\sin\theta)$

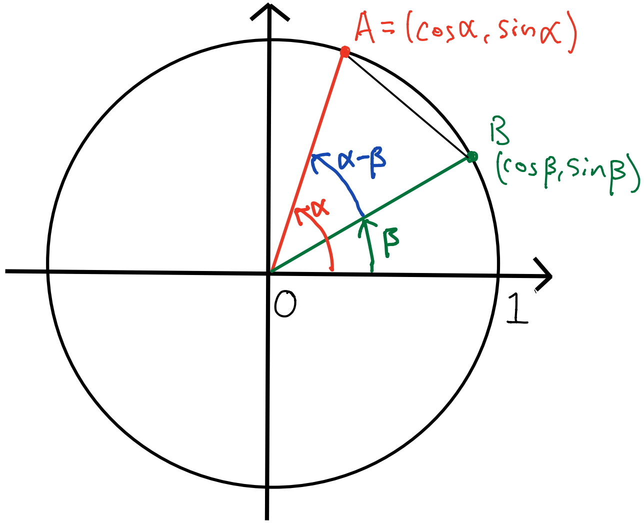
$$\tan(\theta+\pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$ ← Be careful about sign

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$



Pf of $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Consider A, B, O on the unit circle.

By dot product formula,

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = \|\overrightarrow{OA}\|\|\overrightarrow{OB}\| \cos(\alpha-\beta)$$

$$(\cos\alpha, \sin\alpha)\cdot(\cos\beta, \sin\beta) = (1)(1)\cos(\alpha-\beta)$$

$$\Rightarrow \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Other 5 formulas can be deduced from the proved one:

$$\cos(\alpha+\beta) = \cos(\alpha-(-\beta))$$
$$= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2}-\alpha-\beta\right)$$
$$= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$
$$= \sin\alpha\cos\beta + \sin\alpha\cos\beta$$

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta))$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$
$$= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}$$
$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \tan(\alpha+(-\beta))$$
$$= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double / Half Angle Formula

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Take $\alpha = \beta = \theta$

then $\alpha + \beta = 2\theta$

Half Angle Formulas **(alternate form)**

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \sin^2\theta = \frac{1}{2}\left(1-\cos(2\theta)\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right)$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

eg. $\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\Rightarrow \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$

Double Angle Formulas

Important

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

(By $\sin^2\theta + \cos^2\theta = 1$)

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Product to Sum / Sum to Product Formula

Sum and Difference Formulas 1

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$\Longrightarrow$

Product to Sum Formulas 2

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

a $$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$\Longrightarrow$

Sum to Product Formulas 3

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

b $$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

1 $\Rightarrow$ 2

eg Pf of a

$$\text{R.H.S.} = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$= \frac{1}{2}\begin{pmatrix} \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ +\sin\alpha\cos\beta - \cos\alpha\sin\beta \end{pmatrix}$$

$$= \sin\alpha\cos\beta$$

$$= \text{L.H.S.}$$

2 $\Rightarrow$ 3

eg Pf of b

$$\text{R.H.S.} = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$= 2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]$$

$$= \cos\alpha + \cos\beta$$

$$= \text{L.H.S.}$$

eg Find the exact value of the followings

① $\tan 75^\circ = \tan(30^\circ + 45^\circ)$

$$= \frac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$$

$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3} - 1}$$

② $\cos \frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$

③ $\sin\frac{\pi}{24}\sin\frac{7\pi}{24}$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{24} - \frac{7\pi}{24}\right) - \cos\left(\frac{\pi}{24} + \frac{7\pi}{24}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{4}\right) - \cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2} - 1}{4}$$

eg Prove

① $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Pf

$$\begin{aligned}
\text{L.H.S.} &= \sin 3\theta \\
&= \sin(\theta + 2\theta) \\
&= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta \\
&= \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta) \\
&= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta) \\
&= 3\sin\theta - 4\sin^3\theta \\
&= \text{R.H.S.}
\end{aligned}$$

② $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$

Sol

$$\begin{aligned}
\text{R.H.S.} &= \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \\
&= \frac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}} \\
&= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} \\
&= \cos 2\theta \\
&= \text{L.H.S.}
\end{aligned}$$

Ex Prove

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

eg Let $f(x) = \cos x$

Simplify $\dfrac{f(x+h)-f(x)}{h}$

Sol
$$\frac{f(x+h)-f(x)}{h} = \frac{\cos(x+h)-\cos x}{h}$$
$$= \frac{-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$
$$= \frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

Rmk
$$\lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} = \lim_{h\to 0} -\sin\left(x+\frac{h}{2}\right)\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$
$$= -\sin(x)$$

$$\Rightarrow \frac{d}{dx}\cos x = -\sin x$$